GIFT: Geometric Information Field Theory
Blueprint for Formal Verification

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Abstract

This blueprint documents the formal verification of GIFT (Geometric Information Field Theory) in Lean 4. GIFT derives Standard Model parameters from $E_8 \times E_8$ gauge theory compactified on G_2-holonomy manifolds, achieving 0.087% mean deviation across 18 dimensionless predictions with 455+ machine-verified relations.

The document provides mathematical definitions and theorem statements linked to their Lean formalizations, enabling verification of the proof dependencies and progress tracking.

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Chapter 1

Introduction

GIFT (Geometric Information Field Theory) is a framework that derives Standard Model parameters from $E_8 \times E_8$ gauge theory compactified on G_2-holonomy manifolds. This blueprint documents the formal verification in Lean 4, providing:

- Mathematical definitions linked to Lean declarations
- Theorem statements with proof status (proven/axiom)
- Dependency graph for tracking proof progress

The key insight is that the topological invariants of G_2-manifolds (Betti numbers $b_2 = 21$, $b_3 = 77$) combined with exceptional Lie group dimensions determine physical parameters with remarkable precision.

Chapter 2

Foundations: E8 Lattice

The E_8 root system is the largest exceptional simple Lie algebra. We formalize its lattice structure in $\mathbb{R}^8$.

2.1 Euclidean Space Setup

Definition 2.1 (Standard Euclidean Space)**.** Let $\mathbb{R}^8$ denote the 8-dimensional Euclidean space with standard inner product.

Definition 2.2 (Standard Basis)**.** The standard basis vectors e_i for $i \in \{0, \dots, 7\}$ satisfy $\langle e_i, e_j \rangle = \delta_{ij}$.

Theorem 2.3 (Basis Orthonormality)**.** *For all $i, j \in \{0, \dots, 7\}$:*

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.4 (Norm Squared Sum)**.** *For $v \in \mathbb{R}^8$: $\|v\|^2 = \sum_{i=0}^{7} v_i^2$*

Theorem 2.5 (Inner Product Sum)**.** *For $v, w \in \mathbb{R}^8$: $\langle v, w \rangle = \sum_{i=0}^{7} v_i w_i$*

2.2 E8 Lattice Definition

Definition 2.6 (Integer Coordinates)**.** A vector $v \in \mathbb{R}^8$ has *all integer coordinates* if $v_i \in \mathbb{Z}$ for all i.

Definition 2.7 (Half-Integer Coordinates)**.** A vector $v \in \mathbb{R}^8$ has *all half-integer coordinates* if $v_i \in \mathbb{Z} + \frac{1}{2}$ for all i.

Definition 2.8 (Even Sum)**.** A vector v has *even sum* if $\sum_{i=0}^{7} v_i \in 2\mathbb{Z}$.

Definition 2.9 (E8 Lattice)**.** The E_8 lattice consists of all $v \in \mathbb{R}^8$ satisfying either:

1. All coordinates are integers with even sum, or
2. All coordinates are half-integers with even sum

2.3 Lattice Properties

Theorem 2.10 (E8 Inner Product Integral)**.** *For* $v, w \in E_8$*:* $\langle v, w \rangle \in \mathbb{Z}$

Proof. Case analysis on integer/half-integer coordinates with parity arguments. □

Theorem 2.11 (E8 Norm Squared Even)**.** *For* $v \in E_8$*:* $\|v\|^2 \in 2\mathbb{Z}$

Proof. By Lemma **??**, sum of squared integers has same parity as sum. For half-integers, $\sum(n_i + 1/2)^2 = \sum n_i^2 + \sum n_i + 2$, which is even. □

Theorem 2.12 (E8 Closed Under Subtraction)**.** *For* $v, w \in E_8$*:* $v - w \in E_8$

Definition 2.13 (Weyl Reflection)**.** For a root α with $\langle \alpha, \alpha \rangle = 2$, the Weyl reflection is:

$$s_\alpha(v) = v - \langle v, \alpha \rangle \cdot \alpha$$

Theorem 2.14 (Reflection Preserves Lattice)**.** *For* $\alpha, v \in E_8$ *with* $\langle \alpha, \alpha \rangle = 2$*:* $s_\alpha(v) \in E_8$

Proof. Since $\langle v, \alpha \rangle \in \mathbb{Z}$ by Theorem 2.10 and E_8 is closed under integer scaling and subtraction. □

Chapter 3

Foundations: G2 Cross Product

The 7-dimensional cross product is intimately connected to octonion multiplication and defines the G_2 holonomy structure.

3.1 The Fano Plane

Definition 3.1 (Fano Plane Lines)**.** The Fano plane has 7 lines (cyclic triples):

$$\{0,1,3\},\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,0\},\{5,6,1\},\{6,0,2\}$$

Theorem 3.2 (Fano Line Count)**.** *The Fano plane has exactly 7 lines.*

Definition 3.3 (Epsilon Tensor)**.** The structure constants ε_{ijk} for the 7D cross product:

- $\varepsilon_{ijk} = +1$ for (i,j,k) a cyclic permutation of a Fano line
- $\varepsilon_{ijk} = -1$ for anticyclic permutations
- $\varepsilon_{ijk} = 0$ otherwise

3.2 Cross Product Definition

Definition 3.4 (7D Cross Product)**.** For $u, v \in \mathbb{R}^7$, the cross product is:

$$(u \times v)_k = \sum_{i,j} \varepsilon_{ijk}\, u_i\, v_j$$

Theorem 3.5 (Epsilon Antisymmetry)**.** *For all i,j,k: $\varepsilon_{ijk} = -\varepsilon_{jik}$*

3.3 Cross Product Properties

Theorem 3.6 (Cross Product Bilinearity)**.** *The cross product is bilinear:*

$$(au + v) \times w = a(u \times w) + v \times w \tag{3.1}$$

$$u \times (av + w) = a(u \times v) + u \times w \tag{3.2}$$

Theorem 3.7 (Cross Product Antisymmetry)**.** $u \times v = -v \times u$

Proof. Follows from $\varepsilon_{ijk} = -\varepsilon_{jik}$ and sum reindexing. □

Corollary 3.8 (Cross Self Vanishes)**.** $u \times u = 0$

3.4 Lagrange Identity for 7D Cross Product

Definition 3.9 (Epsilon Contraction)**.** $\sum_k \varepsilon_{ijk}\varepsilon_{lmk}$

Definition 3.10 (Coassociative 4-form)**.** The 7D correction to the Kronecker formula:

$$\psi_{ijlm} = \sum_k \varepsilon_{ijk}\varepsilon_{lmk} - (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$$

Lemma 3.11 (Psi Antisymmetry)**.** $\psi_{ijlm} = -\psi_{ljim}$ *(verified for all* $7^4 = 2401$ *index combinations)*

Lemma 3.12 (Psi Contraction Vanishes)**.** $\sum_{i,j,l,m} \psi_{ijlm}\, u_i u_l v_j v_m = 0$

Proof. Antisymmetric tensor ψ contracted with symmetric $u_i u_l$ vanishes. □

Theorem 3.13 (Lagrange Identity)**.**

$$\|u \times v\|^2 = \|u\|^2\|v\|^2 - \langle u, v\rangle^2$$

Proof. Expand $\|u \times v\|^2$ via coordinate sums. The ε-contraction decomposes into Kronecker deltas plus ψ_{ijlm} terms. By antisymmetry of ψ (verified for all 2401 cases), the ψ-terms vanish under symmetric contraction $u_i u_l v_j v_m$. The Kronecker terms yield $\|u\|^2\|v\|^2 - \langle u, v\rangle^2$. □

Chapter 4

Algebraic Foundations

4.1 Octonion Structure

Definition 4.1 (Imaginary Octonion Count)**.** The octonions $\mathbb{O}$ have 7 imaginary units: $|\mathrm{Im}(\mathbb{O})| = 7$

Definition 4.2 (G2 Dimension)**.** $\dim(G_2) = 14$

4.2 Betti Numbers from Octonions

Definition 4.3 (Second Betti Number)**.** $b_2 = \binom{7}{2}$ (pairs of imaginary octonion units)

Theorem 4.4 (b2 Value)**.** $b_2 = 21$

Definition 4.5 (E7 Fundamental)**.** $\mathrm{fund}(E_7) = 56$

Theorem 4.6 (E7 Decomposition)**.** $\mathrm{fund}(E_7) = 2 \cdot b_2 + \dim(G_2) = 42 + 14 = 56$

Definition 4.7 (Third Betti Number)**.** $b_3 = 3 \cdot b_2 + \dim(G_2)$

Theorem 4.8 (b3 Value)**.** $b_3 = 77$

Theorem 4.9 (b3 from E7)**.** $b_3 = b_2 + \mathrm{fund}(E_7) = 21 + 56 = 77$

Definition 4.10 (H-star)**.** $H^* = b_2 + b_3 + 1 = 99$

Chapter 5

SO(16) Decomposition

The decomposition $E_8 \supset \mathrm{SO}(16)$ reveals how GIFT topological invariants encode gauge bosons and fermions separately.

5.1 SO(n) Dimension

Definition 5.1 (SO(n) Dimension)**.** $\dim(\mathrm{SO}(n)) = \frac{n(n-1)}{2}$

Theorem 5.2 (SO(16) = 120)**.** $\dim(\mathrm{SO}(16)) = \frac{16\times15}{2} = 120$

Theorem 5.3 (SO(7) = b2)**.** $\dim(\mathrm{SO}(7)) = \frac{7\times6}{2} = 21 = b_2$

5.2 Spinor Representation

Definition 5.4 (SO(16) Spinor)**.** The chiral spinor of SO(16) has dimension $2^8/2 = 128$.

Theorem 5.5 (Spinor from Octonions)**.** $2^{|\mathrm{Im}(\mathbb{O})|} = 2^7 = 128$

5.3 Geometric and Spinorial Parts

Definition 5.6 (Geometric Part)**.** The geometric part encodes K_7 topology:

$$\mathrm{geom} = b_2 + b_3 + \dim(G_2) + \mathrm{rank}(E_8) = 21 + 77 + 14 + 8$$

Theorem 5.7 (Geometric = SO(16))**.** $b_2 + b_3 + \dim(G_2) + \mathrm{rank}(E_8) = 120 = \dim(\mathrm{SO}(16))$

Definition 5.8 (Spinorial Part)**.** The spinorial part: $2^{|\mathrm{Im}(\mathbb{O})|} = 128$

Theorem 5.9 (Spinorial = 128)**.** *The spinorial part equals the* SO(16) *spinor dimension.*

5.4 Master Decomposition

Theorem 5.10 (E8 = SO(16) + Spinor)**.**

$$\dim(E_8) = 248 = 120 + 128 = \mathrm{geom} + \mathrm{spin}$$

Theorem 5.11 (Gauge-Fermion Split)**.** *Physical interpretation:*

- *120 = topology + holonomy + Cartan* $\rightarrow$ ***gauge bosons***
- *128 =* 2^7 *from octonions* $\rightarrow$ ***fermions***

Chapter 6

Physical Relations

6.1 Weinberg Angle

The weak mixing angle θ_W is one of the most precisely measured parameters in the Standard Model. GIFT derives an *exact* prediction.

Definition 6.1 (Weinberg Numerator)**.** The numerator is $b_2 = 21$.

Definition 6.2 (Weinberg Denominator)**.** The denominator is $b_3 + \dim(G_2) = 77 + 14 = 91$.

Theorem 6.3 (Exact Weinberg Angle)**.**

$$\sin^2\theta_W = \frac{b_2}{b_3 + \dim(G_2)} = \frac{21}{91} = \frac{3}{13}$$

Proof. Cross-multiplication: $21 \times 13 = 273 = 3 \times 91$. □

Theorem 6.4 (Weinberg Simplified)**.** $\frac{3}{13} = 0.230769\ldots$ *vs experimental* 0.23122 ± 0.00004 *(deviation: 0.19%).*

6.2 Koide Formula

The Koide formula relates the masses of charged leptons. It remained unexplained for 43 years until GIFT derived it from topology.

Definition 6.5 (Koide Numerator)**.** The numerator is $\dim(G_2) = 14$.

Definition 6.6 (Koide Denominator)**.** The denominator is $b_2 = 21$.

Theorem 6.7 (Koide Formula)**.**

$$Q_{\text{Koide}} = \frac{\dim(G_2)}{b_2} = \frac{14}{21} = \frac{2}{3}$$

Proof. Cross-multiplication: $14 \times 3 = 42 = 21 \times 2$. □

Remark 6.8 (Historical Context). The Koide formula $Q = 2/3$ was discovered empirically in 1981 and remained unexplained for 43 years. GIFT derives it in two lines from topology.

6.3 Fine Structure Constant

Definition 6.9 (Algebraic Component). $\alpha_{\text{alg}}^{-1} = \frac{\dim(E_8)+\text{rank}(E_8)}{2} = \frac{248+8}{2} = 128$

Definition 6.10 (Bulk Component). $\alpha_{\text{bulk}}^{-1} = \frac{H^*}{D_{\text{bulk}}} = \frac{99}{11} = 9$

Theorem 6.11 (Fine Structure Base). $\alpha_{\text{base}}^{-1} = 128 + 9 = 137$

Theorem 6.12 (Fine Structure Complete). *With torsion correction:*

$$\alpha^{-1} = \frac{267489}{1952} = 137.033...$$

(experimental: 137.035999...*, deviation: 0.002%)*

6.4 Strong Coupling

Definition 6.13 (Strong Coupling Denominator). $\dim(G_2) - p_2 = 14 - 2 = 12$

Theorem 6.14 (Strong Coupling Structure). $\alpha_s = \frac{\sqrt{2}}{12}$*, where* $12 = \dim(G_2) - p_2$

6.5 Lepton Mass Ratios

Theorem 6.15 (Tau/Electron Ratio).

$$\frac{m_\tau}{m_e} = \dim(K_7) + 10 \times \dim(E_8) + 10 \times H^* = 7 + 2480 + 990 = 3477$$

Theorem 6.16 (Tau/Electron Factorization). $3477 = 3 \times 19 \times 61 = N_{\text{gen}} \times p_8 \times \kappa_T^{-1}$

6.6 Higgs Quartic

Definition 6.17 (Higgs Numerator). λ_H^2 numerator: $\dim(G_2) + 3 = 17$

Theorem 6.18 (Higgs Quartic Coupling).

$$\lambda_H^2 = \frac{17}{1024} \implies \lambda_H = \frac{\sqrt{17}}{32} \approx 0.129$$

6.7 Cosmological Parameters

Theorem 6.19 (Spectral Index Indices). *The spectral index* $n_s = \zeta(11)/\zeta(5)$ *uses:*

- $11 = D_{\text{bulk}}$ *(M-theory dimension)*
- $5 =$ *Weyl factor*

Theorem 6.20 (Dark Energy Fraction). $\Omega_{DE} = \ln(2) \times \frac{98}{99} = \ln(2) \times \frac{H^*-1}{H^*}$

Chapter 7

Joyce Existence Theorem

Joyce's perturbation theorem proves K_7 admits torsion-free G_2 structure.

7.1 PINN Verification

Definition 7.1 (Joyce Threshold). $\varepsilon_0 = 0.0288$ (scaled: 288)

Definition 7.2 (PINN Torsion). $\|T(\varphi_0)\| = 0.00141$ (scaled: 141)

Theorem 7.3 (Below Threshold). $\|T(\varphi_0)\| < \varepsilon_0$ *(20× safety margin)*

7.2 Existence

Theorem 7.4 (K7 Admits Torsion-Free G2). $\exists \varphi : K_7 \to \Omega^3$*, torsion-free* G_2 *structure.*

Theorem 7.5 (Joyce Complete Certificate). *All conditions verified: topological (*$b_2 = 21$*,* $b_3 = 77$*), analytic (contraction mapping), existence.*

Chapter 8

Analytical Metric Extraction

The GIFT-native PINN learns an analytical approximation to the G_2 metric on K_7 by encoding the algebraic structure directly in the neural architecture.

8.1 GIFT-Native PINN Architecture

Definition 8.1 (Standard G2 Form)**.** The standard associative 3-form $\varphi_0 = \sum_{ijk} \varepsilon_{ijk}\, dx^i \wedge dx^j \wedge dx^k$ where ε_{ijk} are the Fano plane structure constants, normalized for $\det(g) = 65/32$.

Definition 8.2 (G2 Adjoint Perturbation)**.** The PINN parameterizes perturbations via the 14-dimensional $\mathfrak{g}_2$ adjoint:

$$\varphi(x) = \varphi_0 + \delta\varphi(x), \quad \delta\varphi \in \mathfrak{g}_2$$

Only 14 functions are learned (not 35).

Theorem 8.3 (Dimension Reduction)**.** *The G2 constraint reduces parameters from 35 to 14:* $35 - 14 = 21 = b_2$

8.2 Certified Bounds

Definition 8.4 (Torsion Bound)**.** PINN torsion bound: $\|T\| < 0.001$

Definition 8.5 (Det Error Bound)**.** Determinant error: $|\det(g) - 65/32| < 10^{-6}$

Theorem 8.6 (Joyce Condition)**.** *The PINN torsion is well below Joyce threshold:* $0.001 < 0.0288$

Theorem 8.7 (20x Margin)**.** $20 \times \|T\|_{PINN} < \varepsilon_{Joyce}$

8.3 Analytical Extraction

Definition 8.8 (Fourier Coefficients)**.** The trained PINN is evaluated on a grid and FFT identifies dominant modes. Coefficients are rationalized to $\mathbb{Q}$ within tolerance 10^{-8}.

Theorem 8.9 (Target in Interval)**.** *The target value* $65/32$ *lies in the certified interval for* $\det(g)$.

Chapter 9

Explicit G2 Metric

The key discovery: the standard G2 form φ_0 scaled by $c = (65/32)^{1/14}$ is the *exact* analytical solution satisfying GIFT constraints.

9.1 The Standard G2 3-form

Definition 9.1 (Associative 3-form)**.** The standard G2 3-form on $\mathbb{R}^7$:

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

In 0-indexed notation:

$$\varphi_0 = e^{012} + e^{034} + e^{056} + e^{135} - e^{146} - e^{236} - e^{245}$$

Theorem 9.2 (Seven Terms)**.** *φ_0 has exactly 7 non-zero terms (of 35 independent components).*

Definition 9.3 (Signs Pattern)**.** The signs are: $[+1, +1, +1, +1, -1, -1, -1]$

9.2 Linear Index Representation

Definition 9.4 (C(7,3) Components)**.** A 3-form on $\mathbb{R}^7$ has $\binom{7}{3} = 35$ independent components, indexed lexicographically: $(0, 1, 2) \mapsto 0$, $(0, 1, 3) \mapsto 1$, etc.

Theorem 9.5 (Non-zero Indices)**.** *The 7 non-zero indices are:* $\{0, 9, 14, 20, 23, 27, 28\}$

Index	***Triple***	***Sign***
0	$(0, 1, 2)$	$+1$
9	$(0, 3, 4)$	$+1$
14	$(0, 5, 6)$	$+1$
20	$(1, 3, 5)$	$+1$
23	$(1, 4, 6)$	-1
27	$(2, 3, 6)$	-1
28	$(2, 4, 5)$	-1

Theorem 9.6 (Sparsity)**.** *Only* $7/35 = 20\%$ *of components are non-zero.*

9.3 The GIFT Scale Factor

Definition 9.7 (Scale Factor)**.** To achieve $\det(g) = 65/32$, we scale φ_0 by:

$$c = \left(\frac{65}{32}\right)^{1/14} \approx 1.0543$$

Theorem 9.8 (Scaling Derivation)**.** *For $\varphi = c \cdot \varphi_0$ and metric $g_{ij} = \frac{1}{6} \sum_{k,l} \varphi_{ikl}\varphi_{jkl}$:*

1. *Standard φ_0 gives $g = I_7$, so $\det(g) = 1$*
2. *Scaling $\varphi \mapsto c \cdot \varphi$ gives $g \mapsto c^2 \cdot g$*
3. *Therefore $\det(g) \mapsto c^{14} \cdot \det(g)$*
4. *Setting $c^{14} = 65/32$ yields $\det(g) = 65/32$*

9.4 The Explicit Metric

Theorem 9.9 (Scaled Identity Metric)**.** *The induced metric is:*

$$g = c^2 \cdot I_7 = \left(\frac{65}{32}\right)^{1/7} \cdot I_7 \approx 1.1115 \cdot I_7$$

Explicitly:

$$g_{ij} = \begin{cases} (65/32)^{1/7} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Theorem 9.10 (Determinant Verification)**.** $\det(g) = \left[(65/32)^{1/7}\right]^7 = 65/32 = 2.03125$ ***exactly****.*

9.5 Torsion Vanishes

Theorem 9.11 (Zero Torsion)**.** *For a constant 3-form $\varphi(x) = \varphi_0$:*

- *$d\varphi = 0$ (exterior derivative of constant)*
- *$d{*}\varphi = 0$ (same reasoning)*

Therefore $T = 0$ ***exactly****.*

Theorem 9.12 (Joyce Satisfied)**.** $\|T\| = 0 < 0.0288 = \varepsilon_{Joyce}$ *with* infinite *margin.*

9.6 Summary

Theorem 9.13 (Analytical G2 Metric)**.** *The canonical GIFT G2 metric on K_7 is given by:*
3-form *(35 components, 7 non-zero):*

$$\varphi_i = \begin{cases} +c & i \in \{0, 9, 14, 20\} \\ -c & i \in \{23, 27, 28\} \\ 0 & \text{otherwise} \end{cases}$$

where $c = (65/32)^{1/14}$.

Metric *(*7×7 *diagonal):*

$$g = (65/32)^{1/7} \cdot I_7$$

Properties*:*

- $\det(g) = 65/32$ *(exact)*
- $\|T\| = 0$ *(torsion-free)*
- $\mathrm{Hol}(g) = G_2$ *(by construction)*

Remark 9.14 (Simplicity). This is the *simplest possible* G2 structure satisfying GIFT constraints. The solution is a constant 3-form with only 7 non-zero components and a diagonal metric. No PINN training or Fourier analysis is required—the standard G2 form is the answer.

Remark 9.15 (G2 vs Fano). The G2 3-form indices are **different** from Fano plane lines:

$$\begin{array}{rl} \text{G2 3-form:} & (0,1,2),(0,3,4),(0,5,6),(1,3,5),(1,4,6),(2,3,6),(2,4,5) \\ \text{Fano lines:} & (0,1,3),(1,2,4),(2,3,5),(3,4,6),(4,5,0),(5,6,1),(6,0,2) \end{array}$$

Both have 7 terms but represent different structures (3-form vs cross-product).

Chapter 10

169-Parameter Chebyshev-Cholesky Metric

The 169-parameter Chebyshev-Cholesky metric is the first explicit analytical G_2 holonomy metric on a compact TCS 7-manifold. Published as DOI:10.5281/zenodo.18860358.

10.1 Chebyshev-Cholesky Parametrization

Definition 10.1 (Chebyshev Order)**.** The Chebyshev polynomial order $K = 5$, giving $K+1 = 6$ modes.

Definition 10.2 (Cholesky Entries)**.** A 7×7 lower-triangular matrix has $\binom{8}{2} = 28$ independent entries.

Theorem 10.3 (Off-diagonal $= b_2$)**.** *The 21 off-diagonal Cholesky entries equal* $b_2(K_7)$*, reflecting the* b_2*-dimensional moduli space of* G_2 *structures.*

Theorem 10.4 (169 Total Parameters)**.** *Total parameters:* $6 \times 28 + 1 = 169$*, where the* $+1$ *is the ACyl decay rate.*

Theorem 10.5 ($169 = 13^2 = \alpha_{\mathrm{sum}}^2$)**.** *The parameter count is a perfect square:* $169 = (r_{E_8} + W)^2 = 13^2$.

Theorem 10.6 ($168 = |\mathrm{PSL}(2,7)|$)**.** *The Chebyshev parameter count* $168 = |\mathrm{PSL}(2,7)|$*, the Fano automorphism group.*

10.2 SPD Structure and Determinant

Theorem 10.7 ($\det(g) = 65/32$)**.** *The metric satisfies* $\det(g) = 65/32$ *at every evaluation point, with* $\gcd(65,32) = 1$.

Theorem 10.8 (PINN Compression $> 6000\times$)**.** *The PINN model (1,070,471 parameters) compresses to 169 Chebyshev-Cholesky parameters: a ratio of 6,334:1.*

Theorem 10.9 (Index Decomposition)**.** *The 7* K_7 *coordinates decompose as* $1 + 2 + 4 = 7$*: 1 neck parameter, 2* S^1 *fiber angles, 4 K3 surface coordinates.*

Theorem 10.10 (Explicit G_2 Metric Certificate)**.** *Bundled certificate: 12 structural properties of the 169-parameter metric.*

10.3 Newton-Kantorovich Certification

Definition 10.11 (NK Bounds)**.** The NK contraction parameter: $h \leq 665/10^{10} = 6.65 \times 10^{-8}$.

Theorem 10.12 ($h < 1/2$ (Unconditional))**.** *$h \times 2 < 10^{10}$, establishing $h < 1/2$ with safety margin > 7.5 million.*

Theorem 10.13 (Joyce Convergence: 5 Steps)**.** *5 Joyce iteration steps (= Weyl factor) reduce torsion by $\times 2995$.*

Theorem 10.14 (Proximity < 5 ppm)**.** *The certified metric g_0 differs from the true torsion-free metric g^* by at most 4.86×10^{-6} (relative).*

Theorem 10.15 (Final Torsion Below Joyce Threshold)**.** *After 5 steps: $\|T_5\| = 2.98 \times 10^{-5} \ll 0.0288 = \varepsilon_{Joyce}$.*

Definition 10.16 (NK Certificate Structure)**.** Bundled record: n_{params}, h, steps, reduction, proximity with built-in proof of $h < 1/2$.

Theorem 10.17 (NK Master Certificate)**.** *Bundled certificate: 7 structural properties of the NK certification chain.*

10.4 K3 Harmonic Correction and Torsion Classes

Definition 10.18 (G_2 Torsion Decomposition)**.** Under G_2 representation theory: $T \in W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}$ with dimensions $1 + 7 + 14 + 27 = 49 = \dim(K_7)^2$.

Theorem 10.19 (Torsion Space $= \dim(K_7)^2$)**.** $1 + 7 + 14 + 27 = 49 = 7^2$.

Theorem 10.20 (τ_3 Dominates ($> 99\%$))**.** *The W_{27} (symmetric traceless) component carries 99.6% of total torsion.*

Theorem 10.21 ($|d\varphi|^2/|d{*}\varphi|^2 = 1/\mathrm{Weyl}$)**.** *The exterior derivative norm ratio is exactly $1/5 = 1/W$.*

Theorem 10.22 (K3 Contribution $< 0.1\%$)**.** *The K3 fiber contributes only 0.07% of total torsion, verified over 220,000 evaluation points.*

Theorem 10.23 ($|\varphi|^2 = 2b_2 = 42$)**.** *The proper normalization of the associative 3-form.*

Theorem 10.24 ($|{*}\varphi|^2 = |\mathrm{PSL}(2,7)| = 168$)**.** $|{*}\varphi|^2 = 4 \times 42 = 168 = |\mathrm{PSL}(2,7)|$.

Theorem 10.25 (K3 Harmonic Correction Certificate)**.** *Bundled certificate: 10 structural properties of the torsion reduction chain.*

10.5 K3 Newton-Kantorovich Certificate (CI(2,2,2))

Independent Newton–Kantorovich certificate for the K3 surface $\mathrm{CI}(2,2,2) \subset \mathbb{P}^5$ (the K3 fiber of the TCS G_2 construction), via the Donaldson algebraic section method at degree $k = 4$ (126 sections, 31,752 parameters). Two independent β sources both certify $h < 1/2$: a graph-Laplacian bound ($\beta_{\text{Lap}} = 5.66$, $h = 7.83 \times 10^{-2}$, margin $\times 6.4$) and a Jacobian pseudoinverse bound ($\beta_{\text{Jac}} = 2.25$ at $k = 3$, $h = 0.188$, margin $\times 2.7$). The $k = 2$ Jacobian variant fails ($h = 1.55 > 1/2$), confirming sensitivity to ansatz quality.

Definition 10.26 (K3 NK Certificate structure). Structure recording a K3 NK certificate: Donaldson degree, parameter count, honest η_{L^2} (held-out test set), and two independent h-bounds with contraction proofs.

Definition 10.27 (CI(2,2,2) certificate instance). Concrete certificate: $k = 4$, $\eta_{L^2} = 1.596 \times 10^{-2}$, $h_{\mathrm{Lap}} = 7.83 \times 10^{-2}$, $h_{\mathrm{Jac}} = 1.88 \times 10^{-1}$.

Theorem 10.28 (K3 Laplacian source passes). *$h_{\mathrm{Lap}} < 1/2$: the graph-Laplacian β source certifies NK contraction.*

Theorem 10.29 (K3 Jacobian source passes). *$h_{\mathrm{Jac}} < 1/2$: the pseudoinverse β source certifies NK contraction.*

Theorem 10.30 (K3 certificate is selective). *The $k = 2$ Jacobian variant fails ($h = 1.55 > 1/2$), confirming the certificate is sensitive to ansatz quality.*

Theorem 10.31 (K3 NK master certificate). *Six-conjunct master certificate: both β sources pass, $k = 2$ fails, $\eta_{L^2} < 2/100$, parameter count $> 10{,}000$, and $\delta^{\mathrm{cert}}_{K3} < \varepsilon_0 = 0.1$ (Joyce threshold).*

10.6 Collar Re-summation and Indicial Parity

Weighted-analysis estimates for the collar region of the K3-fibered G_2 metric involve the binomial expansion of $(w - \rho)^{3/2}$ in the fiber coordinate, whose order-k coefficient is the generalized binomial coefficient $\binom{3/2}{k} = \frac{3}{2}(\frac{3}{2} - 1)\cdots(\frac{3}{2} - k + 1)/k!$. Two purely analytic facts control how the per-order bounds aggregate: the absolute series sums to exactly 3 (the collar/bulk erosion factor), and the indicial constant is invariant under the weight reflection $\beta \mapsto -\beta$. Both are machine-checked below, the first by an elementary telescoping route that avoids Abel's theorem.

Definition 10.32 (Generalized binomial coefficient). $\mathrm{C32}(k) = \binom{3/2}{k} = (\prod_{i<k}(3/2 - i))/k!$.

Lemma 10.33 (Alternating telescope). *For all n, $\sum_{k=0}^{n}(-1)^k\binom{3/2}{k} = (-1)^n\binom{1/2}{n}$. Combined with $|\binom{1/2}{n}| \le 1/(2n-1)$, this is the elementary engine for the two main results, used in place of any boundary-limit theorem.*

Theorem 10.34 (Absolute re-summation $\sum|\binom{3/2}{k}| = 3$). *The absolute generalized-binomial series is summable with $\sum_{k\ge 0}|\binom{3/2}{k}| = 3$: the collar bound erodes the bulk bound by exactly the factor 3, independently of the adiabatic parameter.*

Theorem 10.35 (Total absolute sum). *$\sum'_k|\binom{3/2}{k}| = 3$.*

Theorem 10.36 (Alternating value without Abel). *$\sum_{k\ge 0}(-1)^k\binom{3/2}{k} = 0$, i.e. the binomial series of $(1+x)^{3/2}$ evaluated at $x = -1$, obtained here without any boundary-limit theorem.*

Definition 10.37 (Indicial polynomial). $P_m(\beta) = \beta^2 - m^2$.

Theorem 10.38 (Indicial parity $K_{\mathrm{ind}}(-1) = K_{\mathrm{ind}}(+1) = 4/3$). *Since P_m is even in β, the indicial constant $K_{\mathrm{ind}}(\beta) = 1/|P_{1/2}(\beta)|$ satisfies $K_{\mathrm{ind}}(-1) = K_{\mathrm{ind}}(+1) = 4/3$: the weighted-Schauder constant carries verbatim across the $\beta = +1$ and $\beta = -1$ weights.*

Chapter 11

TCS Spectral Bounds

Twisted Connected Sum (TCS) manifolds provide a concrete construction for G_2-holonomy manifolds. This chapter formalizes the spectral theory connecting neck geometry to the bare algebraic ratio $\dim(G_2)/H^* = 14/99$ (topological invariant).

11.1 TCS Manifold Structure

Definition 11.1 (TCS Manifold)**.** A TCS manifold $K = M_1 \cup_N M_2$ consists of:

- Two asymptotically cylindrical manifolds M_1, M_2 (building blocks)
- A cylindrical neck $N \cong Y \times [0, L]$ with cross-section Y
- Neck length parameter $L > 0$
- Volume normalization: $\mathrm{Vol}(K) = 1$

Definition 11.2 (TCS Hypotheses Bundle)**.** The six hypotheses (H1)–(H6) for spectral bounds:

(H1) Volume normalization: $\mathrm{Vol}(K) = 1$

(H2) Bounded neck volume: $v_0 \leq \mathrm{Vol}(N) \leq v_1$

(H3) Product metric: $g|_N = dt^2 + g_Y$

(H4) Block Cheeger bound: $h(M_i \setminus N) \geq h_0$

(H5) Balanced blocks: $\mathrm{Vol}(M_i) \in [1/4, 3/4]$

(H6) Neck minimality: $\mathrm{Area}(\Gamma) \geq \mathrm{Area}(Y)$

11.2 Spectral Bound Constants

Definition 11.3 (Lower Bound Constant)**.** $c_1 = v_0^2$ (from Cheeger inequality)

Definition 11.4 (Upper Bound Constant)**.** $c_2 = \frac{16v_1}{1-v_1}$ (from Rayleigh quotient)

Definition 11.5 (Threshold Length)**.** $L_0 = \frac{2v_0}{h_0}$ (neck dominates for $L > L_0$)

11.3 Model Theorem

Theorem 11.6 (TCS Spectral Bounds)**.** *For TCS manifold K with neck length $L > L_0$ satisfying (H1)–(H6):*

$$\frac{c_1}{L^2} \leq \lambda_1(K) \leq \frac{c_2}{L^2}$$

Theorem 11.7 (Inverse Square Scaling)**.** *The spectral gap scales as $\lambda_1 = \Theta(1/L^2)$ as $L \to \infty$.*

Theorem 11.8 (Typical Parameter Values)**.** *For typical TCS with $v_0 = v_1 = 1/2$, $h_0 = 1$:*

- $c_1 = 1/4$
- $c_2 = 16$
- $L_0 = 1$
- *Bound ratio:* $c_2/c_1 = 64$

11.4 Literature Axioms (Langlais/CGN)

Definition 11.9 (Cross-Section)**.** Cross-section of TCS cylindrical end with dimension and Betti numbers.

Definition 11.10 (K3 x S1 Cross-Section)**.** Standard Kovalev construction with dim = 5 and $b_0 = 1, b_1 = 1, b_2 = 22, b_3 = 22, b_4 = 23, b_5 = 1$.

Theorem 11.11 (Langlais Spectral Density)**.** *For TCS family (M_T, g_T) with cross-section X:*

$$\Lambda_q(s) = 2(b_{q-1}(X) + b_q(X))\sqrt{s} + O(1)$$

Reference: Langlais 2024, Comm. Math. Phys.

Theorem 11.12 (K3 x S1 Density Coefficients)**.** *For K3 $\times$ S^1:*

- *2-forms:* $2(b_1 + b_2) = 2(1 + 22) = 46$
- *3-forms:* $2(b_2 + b_3) = 2(22 + 22) = 88$

Theorem 11.13 (CGN No Small Eigenvalues)**.** *For TCS manifold with neck length L: $\exists c > 0$ such that no eigenvalues in $(0, c/L)$. Reference: Crowley-Goette-Nordstrom 2024, Inventiones Math.*

Theorem 11.14 (GIFT Prediction Structure)**.** *The bare ratio $14/99 = \dim(G_2)/H^*$ is consistent with TCS bounds: $1/100 < 14/99 < 1/4$.*

11.5 Connection to Mass Gap

Theorem 11.15 (TCS Bounds Certificate)**.** *Complete verification of TCS spectral bounds and GIFT prediction compatibility.*

Remark 11.16 (Three-Level Derivation). The physical spectral gap $\lambda_1 = 13/99$ emerges from three levels:

- **Model Theorem** (PROVEN): $\lambda_1 \sim 1/L^2$ from TCS geometry

- **Canonical Length** (CONJECTURE): $L^2 \sim H^* = 99$ from volume principle
- **Holonomy Coefficient** (PROVEN): $\dim(G_2) - h = 14 - 1 = 13$ (Berger)

The bare ratio 14/99 receives a correction of $-1/99 = -h/H^*$ from the parallel spinor (see Chapter 12).

Chapter 12

Physical Spectral Gap

The corrected spectral-holonomy identity $\lambda_1 \times H^* = \dim(G_2) - h$ accounts for parallel spinors from the Berger classification. For G_2 holonomy, $h = 1$ (one parallel spinor), giving the physical spectral gap $\lambda_1 = 13/99$ instead of the bare algebraic value 14/99. All results in this chapter are *fully proven* with zero axioms.

12.1 Parallel Spinors (Berger Classification)

Definition 12.1 (G2 Parallel Spinors)**.** G_2-holonomy manifolds admit exactly $h = 1$ parallel spinor (Berger classification; Joyce 2000, Table 10.1).

Definition 12.2 (SU(3) Parallel Spinors)**.** SU(3)-holonomy manifolds (Calabi-Yau 3-folds) admit $h = 2$ parallel spinors.

Theorem 12.3 (Physical Spectral Product)**.** $\dim(G_2) - h = 14 - 1 = 13.$

12.2 Axiom-Free Derivation of 13/99

Definition 12.4 (Physical Gap Numerator)**.** The physical mass gap numerator: $\dim(G_2) - h = 14 - 1 = 13$.

Definition 12.5 (Physical Gap Ratio)**.** The physical mass gap ratio: $\lambda_1 = 13/99$.

Theorem 12.6 (Physical Gap from Topology)**.** $13/99 = (\dim(G_2) - h)/H^*$ *where all quantities are topological.*

Theorem 12.7 (Irreducibility)**.** $\gcd(13, 99) = 1$. *The ratio* 13/99 *is in lowest terms.*

Theorem 12.8 (Spectral-Holonomy Identity (Corrected))**.** $(13/99) \times 99 = 13 = \dim(G_2) - h$.

Theorem 12.9 (Bare to Physical Correction)**.** $14/99 - 13/99 = 1/99 = h/H^*$. *The correction is exactly the parallel spinor count divided by the total cohomological dimension.*

Theorem 12.10 (Cross-Holonomy: SU(3))**.** *For* SU(3) *holonomy:* $\dim(\mathrm{SU}(3)) - h = 8 - 2 = 6$. *Numerically validated:* $\lambda_1 \times H^* = 5.996$ *on* $T^6/\mathbb{Z}_3$.

Theorem 12.11 (Pell Equation)**.** $99^2 - 50 \times 14^2 = 1$. *The bare topological constants* $(14, 99)$ *satisfy a Pell equation with discriminant* $50 = 2 \times 5^2$.

Theorem 12.12 (Physical Spectral Gap Certificate)**.** *Master certificate: 11 properties verified, zero axioms.*

Chapter 13

G_2 Metric Properties

Properties of the numerical G_2 metric on K_7, computed via physics-informed neural networks (PINNs) and compressed to a single 7×7 symmetric positive-definite matrix.

13.1 Non-Flatness of K_7 (Bieberbach Bound)

Definition 13.1 (Bieberbach Bound)**.** For compact flat n-manifolds (Bieberbach, 1911), the third Betti number satisfies $b_3 \leq \binom{n}{3}$, with equality realized by the n-torus. For $n = 7$: $\binom{7}{3} = 35$.

Theorem 13.2 (Bieberbach Bound from Binomial Coefficient)**.** $\binom{7}{3} = 35$.

Theorem 13.3 (Non-Flatness of K_7)**.** $b_3(K_7) = 77 > 35 = \binom{7}{3}$*, so* K_7 *does not admit a flat Riemannian metric.*

Theorem 13.4 (Bieberbach Gap)**.** $b_3 - \binom{7}{3} = 77 - 35 = 42 = 2b_2$.

13.2 Spectral Degeneracy Pattern $[1, 10, 9, 30]$

The first four eigenvalue multiplicities of the Laplacian on K_7 are observed at 5.8σ significance.

Definition 13.5 (Spectral Multiplicities)**.** The first four eigenvalue multiplicities: $[m_1, m_2, m_3, m_4] = [1, 10, 9, 30]$.

Theorem 13.6 (Topological Origin of $m_2 = 10$)**.** $m_2 = 10 = b_2(\mathrm{CI}(2, 2, 2))$*, the second Betti number of the complete intersection building block.*

Theorem 13.7 (Mode Decomposition of $m_4 = 30$)**.** $m_4 = 30 = 13 + 12 + 4 + 1$ *(K3 + fiber + neck + twist modes).*

Theorem 13.8 (Total Modes are Topological)**.** $m_1 + m_2 + m_3 + m_4 = 50 = b_3 - \dim(J_3(\mathbb{O}))$.

13.3 SPD_7 Metric Parametrization

Definition 13.9 (SPD_7 Dimension)**.** The space of 7×7 symmetric positive-definite matrices has $\dim(\mathrm{SPD}_7) = 7 \cdot 8/2 = 28$ independent entries.

Theorem 13.10 (SPD$_7$ from Formula)**.** $\dim(K_7) \cdot (\dim(K_7) + 1)/2 = 28$*.*

Theorem 13.11 (SPD$_7$ = Twice Holonomy)**.** $28 = 2 \times \dim(G_2) = 2 \times 14$*: the metric degrees of freedom are twice the holonomy dimension.*

Theorem 13.12 (PINN Compression Ratio)**.** 1,070,471/28 = 38,231*: compression from PINN parameters to the analytical metric matrix.*

13.4 Triple Derivation of $\det(g) = 65/32$

Three algebraically independent paths yield the same determinant.

Theorem 13.13 (Path 1: Weyl Formula)**.** $\mathrm{Weyl} \times (\mathrm{rank}(E_8) + \mathrm{Weyl}) = 5 \times 13 = 65$ *and* $2^{\mathrm{Weyl}} = 2^5 = 32$*.*

Theorem 13.14 (Path 2: Cohomological Formula)**.** $p_2 \times (b_2 + \dim(G_2) - N_{\mathrm{gen}}) + 1 = 2 \times 32 + 1 = 65$ *and* $b_2 + \dim(G_2) - N_{\mathrm{gen}} = 21 + 14 - 3 = 32$*.*

Theorem 13.15 (Path 3: H^* Formula)**.** $H^* - b_2 - \alpha_{\mathrm{sum}} = 99 - 21 - 13 = 65$*.*

Theorem 13.16 (Triple Consistency)**.** *All three paths yield* $\det(g) = 65/32$*.*

Theorem 13.17 (Irreducibility)**.** $\gcd(65, 32) = 1$*: the fraction* 65/32 *is irreducible.*

13.5 $\kappa_T^{-1} = 61$ Structural Decomposition

Theorem 13.18 (κ_T from F_4 and Generations)**.** $\kappa_T^{-1} = 61 = \dim(F_4) + N_{\mathrm{gen}}^2 = 52 + 9$*.*

Theorem 13.19 (κ_T^{-1} is Prime)**.** 61 *is prime.*

Theorem 13.20 (κ_T to τ-Electron Mass Ratio)**.** $(\kappa_T^{-1} + 1) \times (b_3 - b_2) + \mathrm{Weyl} = 62 \times 56 + 5 = 3477 = m_\tau/m_e$*.*

13.6 Master Certificate

Theorem 13.21 (G_2 Metric Properties Certificate)**.** *All 12 conjuncts of the G_2 metric master certificate are proven: non-flatness, spectral degeneracies, SPD$_7$ parametrization,* $\det(g) = 65/32$ *triple derivation, and κ_T decomposition.*

Chapter 14

TCS Piecewise Metric Structure

On the TCS manifold $K_7 = M_1 \cup M_2$, the analytical metric is piecewise-constant:

$$g(t) = \begin{cases} G & t < 3/4 \quad \text{(quintic block)} \\ J^\top G J & t > 3/4 \quad \text{(CI block, Kovalev twist)} \end{cases}$$

where J is an involutory orthogonal matrix from the automorphism group of the Fano plane $\mathrm{PG}(2,2)$.

14.1 Building Block Asymmetry

Definition 14.1 (Building blocks)**.** M_1 is the quintic in $\mathbb{CP}^4$ with $b_2 = 11$, $b_3 = 40$. M_2 is the complete intersection $\mathrm{CI}(2,2,2)$ in $\mathbb{CP}^6$ with $b_2 = 10$, $b_3 = 37$.

Theorem 14.2 (b_3 asymmetry equals N_{gen})**.** $b_3(M_1) - b_3(M_2) = 40 - 37 = 3 = N_{\mathrm{gen}}$.

Theorem 14.3 (b_2 asymmetry)**.** $b_2(M_1) - b_2(M_2) = 11 - 10 = 1$.

14.2 Effective Degrees of Freedom per Block

Definition 14.4 ($H^*(M_i)$)**.** $H^*(M_i) := 1 + b_2(M_i) + b_3(M_i)$ for each building block.

Theorem 14.5 ($H^*(M_1) = \dim(F_4)$)**.** *$H^*(M_1) = 1 + 11 + 40 = 52 = \dim(F_4)$, where F_4 is the automorphism group of the exceptional Jordan algebra $J_3(\mathbb{O})$.*

Theorem 14.6 ($H^*(M_2) = h(G_2)\times\mathrm{rank}(E_8)$)**.** *$H^*(M_2) = 1+10+37 = 48 = h(G_2)\times\mathrm{rank}(E_8) = 6 \times 8$, where $h(G_2) = 6$ is the Coxeter number of G_2.*

Theorem 14.7 (Block sum)**.** *$H^*(M_1) + H^*(M_2) = 52 + 48 = 100 = H^* + 1$. The extra 1 accounts for the double-counted b_0 from each connected block.*

14.3 Matrix Space Decomposition

Theorem 14.8 ($\dim(K_7)^2 = 2 \cdot \dim(G_2) + b_2$)**.** *The 7×7 matrix space decomposes as $49 = 28 + 21 = 2 \cdot \dim(G_2) + b_2$. The symmetric part (28 = metric DOF) equals $2 \times \dim(G_2)$; the antisymmetric part (21 = torsion 2-forms) equals b_2.*

14.4 Fano Plane Automorphism Group

Theorem 14.9 ($|\mathrm{PSL}(2,7)| = \mathrm{rank}(E_8) \times b_2$)**.** *The Fano automorphism group has order* $168 = 8 \times 21$.

14.5 Fano Incidence Arithmetic

Theorem 14.10 ($b_2 = \dim(K_7) \times N_{\mathrm{gen}}$)**.** *From the Fano plane: 7 points* $\times$ *3 lines per point* $= 21 = b_2$.

14.6 Kovalev Involution Eigenspace

Theorem 14.11 (Eigenspace split)**.** *The Kovalev twist* J *splits* $\mathbb{R}^7$ *into* $V_+ \oplus V_-$ *with* $\dim(V_+) = N_{\mathrm{gen}} + 1 = 4$ *and* $\dim(V_-) = N_{\mathrm{gen}} = 3$.

Theorem 14.12 ($C(7,4) = C(7,3) = 35$)**.** *The number of Kovalev-type involutions (with 4 fixed directions) equals the dimension of* $\Lambda^3(\mathbb{R}^7)$*: both equal 35.*

Theorem 14.13 ($35 + \dim(G_2) = \dim(K_7)^2$)**.** $C(7,3) + 14 = 35 + 14 = 49 = 7^2$.

14.7 Master Certificate

Theorem 14.14 (TCS Piecewise Metric Certificate)**.** *All 10 conjuncts of the TCS piecewise metric master certificate are proven.*

Chapter 15

Conformal Rigidity

The G_2 metric on K_7 is completely determined by two constraints: G_2 holonomy (kills 27 traceless deformations) and $\det(g) = 65/32$ (fixes the remaining conformal modulus), leaving zero free parameters.

15.1 G_2 Representation Decompositions

Theorem 15.1 ($\mathrm{Sym}^2(V_7) = 1 \oplus 27$ under G_2)**.** *The symmetric 2-tensors decompose as conformal trace (*1*) plus traceless part (*$27 = \dim(J_3(\mathbb{O}))$*):* $28 = 1 + 27$*.*

Theorem 15.2 ($\Lambda^2(V_7) = 7 \oplus 14$ under G_2)**.** *The 2-forms decompose as standard (*7*) plus adjoint (*14*):* $21 = \dim(K_7) + \dim(G_2)$*.*

Theorem 15.3 ($\mathrm{End}(V_7) = 1 \oplus 7 \oplus 14 \oplus 27$)**.** *The endomorphism algebra uses all four fundamental* G_2 *representations:* $49 = 1 + 7 + 14 + 27 = \dim(K_7)^2$*.*

Theorem 15.4 ($\Lambda^3(V_7) = 1 \oplus 7 \oplus 27$)**.** *The 3-forms decompose as* $C(7, 3) = 35 = 1 + \dim(K_7) + \dim(J_3(\mathbb{O}))$*. The singlet is the associative 3-form* φ_0*; the* 27 *is the same representation as in* $\mathrm{Sym}^2_0(V_7)$*.*

15.2 Conformal Rigidity Theorem

Theorem 15.5 (Zero residual degrees of freedom)**.** *The metric has* $28 - 27 - 1 = 0$ *free parameters:*

1. $\dim(\mathrm{SPD}_7) = 28$ *total metric DOF*
2. G_2 *holonomy kills* $\dim(J_3(\mathbb{O})) = 27$ *traceless directions*
3. $\det(g) = 65/32$ *fixes the remaining conformal modulus*

Theorem 15.6 (Conformal exponent $= \dim(G_2)$)**.** *For the isotropic metric* $g = c^2 \cdot I_7$*:* $\det(g) = c^{2\times\dim(K_7)} = c^{\dim(G_2)}$*. The holonomy dimension determines the conformal equation.*

Theorem 15.7 ($\dim(J_3(\mathbb{O})) = N_{\mathrm{gen}}^3$)**.** $27 = 3^3$*: the traceless symmetric space dimension equals the cube of the generation number.*

15.3 Moduli Space Structure

Theorem 15.8 ($b_3 - b_2 = \dim(\text{fund. } E_7)$)**.** *The moduli gap* $77 - 21 = 56 = \dim(\text{fund. } E_7) = \text{rank}(E_8) \times \dim(K_7) = 8 \times 7$.

15.4 Master Certificate

Theorem 15.9 (Conformal Rigidity Certificate)**.** *All 9 conjuncts of the conformal rigidity certificate are proven.*

Chapter 16

Spectral Scaling on the TCS Neck

The neck $N \cong [0, L] \times Y$ of the TCS manifold K_7 supports a waveguide spectrum. The longitudinal Neumann eigenvalues $\lambda_n = n^2\pi^2/L^2$ connect to GIFT topological constants through remarkable arithmetic identities: the eigenvalue sum $1^2 + 2^2 + 3^2 = \dim(G_2)$, the sub-gap mode count equals N_{gen}, and the neck length ratio $L^2/\pi^2 = H^*/\dim(G_2)$ satisfies a Pell equation.

16.1 Neumann Eigenvalue Hierarchy

Definition 16.1 (Neumann eigenvalue ratio)**.** The n-th Neumann eigenvalue ratio $\lambda_n/\lambda_1 = n^2$, from the 1D problem $-f'' = \lambda f$ on $[0, L]$ with $f'(0) = f'(L) = 0$.

Theorem 16.2 (Second spectral gap = N_{gen})**.** *The gap between the first two excited modes is* $\lambda_2 - \lambda_1 = 3 = N_{\mathrm{gen}}$.

Theorem 16.3 (Third spectral gap = Weyl factor)**.** *The gap* $\lambda_3 - \lambda_2 = 5 = W$.

Theorem 16.4 (Eigenvalue sum = $\dim(G_2)$)**.** $1^2 + 2^2 + 3^2 = 14 = \dim(G_2)$*: the holonomy dimension emerges as a sum of eigenvalue ratios.*

Theorem 16.5 (Eigenvalue sum = $h(E_8)$)**.** $1^2 + 2^2 + 3^2 + 4^2 = 30 = h(E_8)$*: the Coxeter number of* E_8 *emerges at the fourth level.*

16.2 Neck Length Ratio and Selection Constant

Theorem 16.6 (Neck ratio from GIFT constants)**.** $L^2/\pi^2 = H^*/\dim(G_2) = 99/14$.

Theorem 16.7 (Selection constant structure)**.** $\dim(G_2) = p_2 \times \dim(K_7) = 2 \times 7$*: the Pontryagin-manifold factorization of the holonomy dimension.*

Theorem 16.8 ($\kappa \times H^* = L^2/\pi^2$)**.** $(1/14) \times 99 = 99/14$*: the rational skeleton of* $L^2 = \kappa H^*$.

16.3 Division Algorithm Connection

Theorem 16.9 (Euclidean division of H^* by $\dim(G_2)$)**.** $H^* = \dim(K_7) \times \dim(G_2) + 1$, *i.e.,* $99 = 7 \times 14 + 1$. *The quotient is* $\dim(K_7)$ *and the remainder is 1 (the parallel spinor count for* G_2 *holonomy from Berger's classification).*

Theorem 16.10 (Remainder = spinor correction)**.** *H^* mod $\dim(G_2) = 1$, and $(\dim(G_2) - 1)/H^* = 13/99$: the physical spectral gap.*

16.4 Sub-Gap Mode Counting

Theorem 16.11 (Sub-gap threshold = $\dim(K_7)$)**.** *$\lfloor L^2/\pi^2 \rfloor = \lfloor 99/14 \rfloor = 7 = \dim(K_7)$.*

Theorem 16.12 (Sub-gap mode count = N_{gen})**.** *Modes with $n^2 \leq 7$: $n = 0, 1, 2$ (since $3^2 = 9 > 7$). The count is $3 = N_{\text{gen}}$: the three fermion generations correspond to the three longitudinal waveguide modes below the cross-section gap.*

16.5 Second Eigenvalue and E_7

Theorem 16.13 ($\lambda_2 \times H^* = \dim(\text{fund. } E_7)$)**.** *$4 \times \dim(G_2) = 56 = \dim(\text{fund. } E_7) = b_3 - b_2$.*

Theorem 16.14 ($56 = C(\text{rank}(E_8), N_{\text{gen}})$)**.** *$\dim(\text{fund. } E_7) = \binom{8}{3}$: a binomial coefficient involving the rank of E_8 and N_{gen}.*

16.6 Pell Equation

Theorem 16.15 (Pell equation for $(H^*, \dim(G_2))$)**.** *$99^2 - 50 \times 14^2 = 1$. The discriminant $50 = p_2 \times W^2 = 2 \times 5^2$ connects to the Pontryagin class and Weyl factor.*

16.7 Spectral-Topological Dictionary

Theorem 16.16 (Spectral ladder)**.** *$\lambda_1 H^* = 14$, $\lambda_2 H^* = 56$, gap $= 42 = 2b_2$.*

16.8 Master Certificate

Theorem 16.17 (Spectral Scaling Certificate)**.** *All 12 conjuncts of the spectral scaling certificate are proven.*

Chapter 17

Poincaré Duality and the GIFT Spectrum

Poincaré duality doubles the GIFT spectrum: the full Betti sequence $(1, 0, 21, 77, 77, 21, 0, 1)$ of K_7 sums to $198 = 2 \times H^*$. The central structural identity $H^* = 1 + 2 \times \dim(K_7)^2$ shows the effective cohomological dimension is determined by the manifold dimension alone.

17.1 Full Betti Sequence and Total Betti Number

Definition 17.1 (Total Betti number)**.** The total Betti number $\beta_{\text{total}} = \sum_{k=0}^{7} b_k = 198$.

Theorem 17.2 (Poincaré duality doubles H^*)**.** $\beta_{\text{total}} = 2 \times H^* = 198$.

Theorem 17.3 (Total Betti factored)**.** $198 = 2 \times N_{\text{gen}}^2 \times D_{\text{bulk}} = 2 \times 9 \times 11$.

17.2 Structural Identity: $H^* = 1 + 2\dim(K_7)^2$

Theorem 17.4 (Betti pair from manifold dimension)**.** $b_2 + b_3 = 2 \times \dim(K_7)^2 = 98$.

Theorem 17.5 (H^* structural identity)**.** $H^* = 1 + 2 \times \dim(K_7)^2 = 99$.

17.3 Holonomy Embedding Chain $G_2 < \mathrm{SO}(7) < \mathrm{GL}(7)$

Definition 17.6 (Dimension of SO(7))**.** $\dim(\mathrm{SO}(7)) = \binom{7}{2} = 21$.

Theorem 17.7 ($\dim(\mathrm{SO}(7)) = b_2$)**.** *The second Betti number equals the dimension of* $\mathrm{SO}(7)$.

Theorem 17.8 (Codimension $\mathrm{GL}(7)/G_2 = \binom{7}{3}$)**.** $\dim(\mathrm{GL}(7)) - \dim(G_2) = \binom{7}{3} = 35$*: the number of independent 3-forms on* K_7.

Theorem 17.9 (Chain additivity)**.** $\dim(G_2) + \dim(K_7) + 28 = \dim(\mathrm{GL}(7))$*, i.e.,* $14 + 7 + 28 = 49$.

17.4 G_2 Torsion Decomposition

Theorem 17.10 (Torsion space $= \dim(K_7)^2$)**.** $1+\dim(K_7)+\dim(G_2)+\dim(J_3(\mathbb{O})) = \dim(K_7)^2 = 49$.

Theorem 17.11 (Torsion-free constraints $= \binom{7}{3}$)**.** $1 + \dim(K_7) + \dim(J_3(\mathbb{O})) = \binom{7}{3} = 35$.

17.5 SU(3) Branching Rule

Theorem 17.12 (G_2 adjoint branching)**.** $\dim(G_2) = \dim(\mathrm{SU}(3)) + 2N_{\mathrm{gen}} = 8 + 6 = 14$.

17.6 Betti–Torsion Bridge

Theorem 17.13 (Betti pair $= 2\times$ torsion space)**.** $b_2 + b_3 = 2 \times (1 + \dim(K_7) + \dim(G_2) + \dim(J_3(\mathbb{O})))$.

17.7 Master Certificate

Theorem 17.14 (Poincaré Duality Certificate)**.** *All 12 conjuncts of the Poincaré duality certificate are proven.*

Chapter 18

G_2 Differential Geometry

The axiom-free formalization of G_2 differential geometry on $\mathbb{R}^7$: exterior algebra $\Lambda^*(\mathbb{R}^7)$, differential forms $\Omega^k(\mathbb{R}^7)$, Hodge star $\star : \Omega^3 \to \Omega^4$, and the torsion-free condition $d\varphi = 0 \wedge d\psi = 0$.

18.1 Differential Forms on $\mathbb{R}^7$

Definition 18.1 (Differential k-Form)**.** A differential k-form on $\mathbb{R}^7$: position-dependent coefficients $\omega \in C^\infty(\mathbb{R}^7, \mathbb{R}^{\binom{7}{k}})$.

Definition 18.2 (Exterior Derivative)**.** Structure encoding $d : \Omega^k \to \Omega^{k+1}$ with $d^2 = 0$.

Theorem 18.3 (Exact $\Rightarrow$ Closed)**.** *$d\omega = 0 \Rightarrow d(d\omega) = 0$: exactness implies closedness.*

Definition 18.4 (Standard G_2 Form Data)**.** The canonical G_2 3-form φ with Fano plane indices and its dual ψ.

Theorem 18.5 (Standard G_2 Torsion-Free)**.** *The constant G_2 form on flat $\mathbb{R}^7$ is torsion-free: $d\varphi = 0 \wedge d\psi = 0$.*

18.2 Hodge Star on $\mathbb{R}^7$

Theorem 18.6 ($\star\star$ Sign in 7 Dimensions)**.** *For all $k \in \{0, \dots, 7\}$: $(-1)^{k(7-k)} = +1$. In 7 dimensions, the Hodge star satisfies $\star\star = +1$ (not ± 1).*

Definition 18.7 (Hodge Star $\star_3$)**.** $\star : \Omega^3(\mathbb{R}^7) \to \Omega^4(\mathbb{R}^7)$ via complement map and Levi-Civita signs.

Theorem 18.8 ($\psi = \star\varphi$ (Proven))**.** *The G_2 4-form ψ equals the Hodge dual of φ: $\psi = \star\varphi$.* ***Proven**, not axiomatized.*

Theorem 18.9 (Hodge Involutivity)**.** *$\star_4(\star_3(\omega)) = \omega$ for constant 3-forms.*

Definition 18.10 (G_2 Geometric Data)**.** Structure combining φ, ψ, exterior derivative d, and Hodge star $\star$ with the torsion-free condition: $d\varphi = 0 \wedge d(\star\varphi) = 0$.

Theorem 18.11 (Standard G_2 Geometry Torsion-Free)**.** *The standard G_2 structure on flat $\mathbb{R}^7$ is torsion-free.*

18.3 Master Certificate

Theorem 18.12 (Geometry Infrastructure Certificate)**.** *Complete G_2 differential geometry infrastructure with zero axioms: $\varphi \in \Omega^3$, $\psi = \star\varphi$ (proven), $\star\star = +1$ (proven), torsion-free on flat $\mathbb{R}^7$ (proven).*

Chapter 19

Dimensional Hierarchy

The electroweak–Planck hierarchy $M_{\text{EW}}/M_{\text{Pl}} \approx 10^{-17}$ is derived from K_7 topology via:

$$\frac{M_{\text{EW}}}{M_{\text{Pl}}} = e^{-H^*/\text{rank}(E_8)} \times \varphi^{-54} = e^{-99/8} \times \varphi^{-54} \approx 10^{-17}$$

19.1 Cohomological Suppression

Definition 19.1 (Cohomological Ratio)**.** The cohomological exponent $H^*/\text{rank}(E_8) = 99/8$.

Definition 19.2 (Cohomological Suppression)**.** $S_{\text{cohom}} := e^{-H^*/\text{rank}(E_8)} = e^{-99/8}$.

Theorem 19.3 (Suppression Magnitude)**.** $10^{-6} < e^{-99/8} < 10^{-5}$.

Definition 19.4 (Jordan Suppression)**.** $S_{\text{Jordan}} := \varphi^{-54} = (\varphi^{-2})^{27}$, where $27 = \dim(J_3(\mathbb{O}))$ and $\varphi^{-2} \approx 0.382$.

Theorem 19.5 (Jordan Suppression Small)**.** $\varphi^{-54} < 10^{-10}$.

Definition 19.6 (Hierarchy Ratio)**.** $R_{\text{hier}} := S_{\text{cohom}} \times S_{\text{Jordan}} = e^{-99/8} \times \varphi^{-54}$.

Theorem 19.7 (Hierarchy Very Small)**.** $R_{\text{hier}} < 10^{-15}$. *The hierarchy problem is solved by topology.*

Theorem 19.8 (Hierarchy Bounds)**.** $-39 < \ln(R_{\text{hier}}) < -38$, *matching* $M_{\text{EW}}/M_{\text{Pl}} \approx 10^{-17}$.

19.2 Vacuum Structure

Definition 19.9 (Number of Vacua)**.** $N_{\text{vacua}} := b_2 = 21$ associative 3-cycles on K_7.

Theorem 19.10 (Vacua Equal b_2)**.** $N_{\text{vacua}} = b_2 = 21$.

Theorem 19.11 (Moduli Dimension)**.** $\dim(\mathcal{M}) = b_3 = 77$.

Theorem 19.12 (TCS Building Block Decomposition)**.** $b_3 = 40 + 37$ *(quintic and CI blocks).*

Theorem 19.13 (Vacuum–Topology Correspondence)**.** *Complete vacuum structure derived from topology.*

19.3 $E_8 \to E_6$ Symmetry Cascade

Theorem 19.14 (E_6 Fundamental)**.** dim(fund. E_6) $= 27 = \dim(J_3(\mathbb{O}))$.

Theorem 19.15 (E_8–E_6–SU(3) Branching)**.** $248 = 78+8+2\times27\times3 = \dim(E_6)+\dim(\mathrm{SU}(3))+162$.

19.4 Absolute Lepton Masses

Theorem 19.16 (m_τ/m_e Formula)**.** $m_\tau/m_e = (b_3 - b_2)(\kappa_T^{-1} + 1) + W = 56 \times 62 + 5 = 3477$.

Theorem 19.17 (m_τ/m_e Expanded)**.** $3477 = 3472 + 5$ *where* $3472 = 56 \times 62$ *(Betti* $\times$ *kappa) and* $5 = W$ *(Weyl factor).*

Theorem 19.18 (m_τ/m_e Prime Factorization)**.** $3477 = 3 \times 19 \times 61$ *(all prime factors are GIFT-expressible).*

Theorem 19.19 (m_μ/m_e Bounds)**.** $206 < 27^\varphi < 208$ *(Jordan algebra base with golden exponent).*

Theorem 19.20 (y_τ Yukawa)**.** $y_\tau = 1/98 = 1/(H^* - 1)$ *(tau Yukawa from cohomology).*

Theorem 19.21 (Mass Formulas Verified)**.** *All absolute mass formulas verified.*

Chapter 20

Extended Observables

Approximately 50 physical observables derived from topology with zero free parameters. Mean deviation: 0.24%.

20.1 Weak Mixing Angle (14 Expressions)

Definition 20.1 ($\sin^2\theta_W$ Observable)**.** $\sin^2\theta_W := 3/13$.

Theorem 20.2 (14 Equivalent Expressions)**.** *Primary:* $\sin^2\theta_W = N_{\text{gen}}/(\dim(G_2) - 1) = 3/13$. *Deviation:* 0.20% *from experiment.*

20.2 PMNS Neutrino Mixing

Definition 20.3 ($\sin^2\theta_{12}^{\text{PMNS}}$)**.** $\sin^2\theta_{12} := 4/13$ (solar mixing angle).

Definition 20.4 ($\sin^2\theta_{23}^{\text{PMNS}}$)**.** $\sin^2\theta_{23} := 6/11 = 6/D_{\text{bulk}}$ (atmospheric mixing angle).

Definition 20.5 ($\sin^2\theta_{13}^{\text{PMNS}}$)**.** $\sin^2\theta_{13} := 11/496 = D_{\text{bulk}}/(2 \times \dim(E_8))$ (reactor angle).

20.3 Quark Mass Ratios

Definition 20.6 (m_s/m_d)**.** $m_s/m_d := 20$.

Definition 20.7 (m_b/m_t)**.** $m_b/m_t := 1/42 = 1/(2b_2)$ — the structural invariant 42 appears!

Definition 20.8 (m_c/m_s)**.** $m_c/m_s := 246/21 = (\dim(E_8) - p_2)/b_2$.

Definition 20.9 (m_u/m_d)**.** $m_u/m_d := 79/168$.

20.4 Boson Mass Ratios

Definition 20.10 (m_H/m_W)**.** $m_H/m_W := 81/52$.

Definition 20.11 (m_H/m_t)**.** $m_H/m_t := 8/11 = \text{rank}(E_8)/D_{\text{bulk}}$.

Definition 20.12 (m_W/m_Z)**.** $m_W/m_Z := 37/42$.

20.5 CKM Quark Mixing

Definition 20.13 (Cabibbo Angle)**.** $\sin^2\theta_{12}^{\text{CKM}} := 56/248 = \dim(\text{fund. } E_7)/\dim(E_8)$.

Definition 20.14 (Wolfenstein A)**.** $A := 83/99 = 83/H^*$.

Definition 20.15 ($\sin^2\theta_{23}^{\text{CKM}}$)**.** $\sin^2\theta_{23}^{\text{CKM}} := 7/168 = \dim(K_7)/(8 \times b_2)$.

20.6 Cosmological Parameters

Definition 20.16 (Dark Matter / Baryonic Ratio)**.** $\Omega_{\text{DM}}/\Omega_b := 43/8 = (1 + 2b_2)/\text{rank}(E_8)$.

Definition 20.17 (Hubble Parameter)**.** $h := 167/248 = 167/\dim(E_8)$. Deviation: 0.09% from Planck 2018.

Definition 20.18 (σ_8)**.** $\sigma_8 := 17/21 = 17/b_2$.

Definition 20.19 (Primordial Helium Y_p)**.** $Y_p := 15/61 = 15/\kappa_T^{-1}$.

20.7 Universality

Theorem 20.20 (The 42 Universality)**.** *The structural invariant* $2b_2 = 42$ *appears in both particle physics (*$m_b/m_t = 1/42$*) and cosmology (*$\Omega_{\text{DM}}/\Omega_b = 43/8$*).*

Theorem 20.21 (8/11 Universality)**.** $\text{rank}(E_8)/D_{\text{bulk}} = 8/11$ *appears in both* m_H/m_t *and* $\sin^2\theta_{23}^{\text{PMNS}}$.

Chapter 21

Mass Gap Ratio and Universal Law

The algebraic ratio $\dim(G_2)/H^* = 14/99$ is a topological invariant (holonomy dimension divided by cohomological complexity). The analytical mass gap is $\lambda_1 = 6\pi^2/475 \approx 0.12467$, giving $\lambda_1 \cdot H^* \approx 12.34$ (not 14 or 13).

21.1 Spectral Theory Foundations

Definition 21.1 (Compact Manifold)**.** Abstract compact Riemannian manifold with dimension and volume.

Definition 21.2 (Laplace–Beltrami Operator)**.** The Laplace–Beltrami operator Δ on a compact Riemannian manifold.

Definition 21.3 (Mass Gap)**.** $\Delta_{\text{gap}} := \inf \sigma(\Delta) \setminus \{0\}$ (smallest positive eigenvalue of Δ).

Theorem 21.4 (Mass Gap Positivity)**.** $\Delta_{\text{gap}} > 0$ *for compact manifolds.*

21.2 Bare Algebraic Ratio: 14/99

Definition 21.5 (Mass Gap Numerator)**.** $\lambda_{\text{num}} := \dim(G_2) = 14$.

Definition 21.6 (Mass Gap Denominator)**.** $\lambda_{\text{den}} := H^* = 99$.

Definition 21.7 (Mass Gap Ratio)**.** Bare algebraic ratio: $\dim(G_2)/H^* = 14/99$. This is a topological invariant, not the spectral eigenvalue (the analytical mass gap is $\lambda_1 = 6\pi^2/475 \approx 0.12467$).

Theorem 21.8 (Mass Gap Irreducible)**.** $\gcd(14, 99) = 1$*: the fraction* $14/99$ *is irreducible.*

Theorem 21.9 (Mass Gap from Holonomy–Cohomology)**.** *The ratio* $14/99$ *arises as* $\dim(\text{Hol})/(b_2 + b_3 + 1)$ *— holonomy divided by cohomological complexity.*

Theorem 21.10 (Mass Gap Certificate)**.** *Complete mass gap ratio certification.*

21.3 Universal Spectral Law

Theorem 21.11 (Product Formula)**.** $14 = \dim(G_2)$ *and* $99 = H^*$*: the bare ratio components are topological. The analytical spectral product* $\lambda_1 \cdot H^* \approx 12.34$ *(from* $\lambda_1 = 6\pi^2/475$*).*

Theorem 21.12 (Ratio Irreducible)**.** $\gcd(14, 99) = 1$.

Theorem 21.13 (Physical Mass Gap)**.** *Using the bare ratio:* $\Delta = (14/99) \times \Lambda_{\text{QCD}}$ *with* $\Lambda_{\text{QCD}} = 200$ MeV *gives* ≈ 28.3 MeV*. The analytical mass gap* $\lambda_1 = 6\pi^2/475$ *gives* ≈ 24.9 MeV.

Theorem 21.14 (Universal Law Certificate)**.** *Complete universal spectral law certificate.*

Chapter 22

Selection Principle and Yang–Mills

Selection of the canonical neck length $L^* = \sqrt{\kappa H^*}$ with $\kappa = \pi^2/14$, Cheeger inequality, and Kaluza–Klein spectral bridge.

22.1 Selection Constant

Definition 22.1 (κ)**.** $\kappa := \pi^2/\dim(G_2) = \pi^2/14$.

Theorem 22.2 (κ Bounds)**.** *$9/14 < \kappa < 10/14$, i.e., $0.643 < \kappa < 0.714$.*

Theorem 22.3 (Building Blocks Match K_7)**.** *$b_2(M_1) + b_2(M_2) = b_2$ and $b_3(M_1) + b_3(M_2) = b_3$ (Mayer–Vietoris).*

Definition 22.4 (Canonical Neck Length)**.** $L^* := \sqrt{\kappa H^*} = \sqrt{99\pi^2/14}$.

Theorem 22.5 (L^* Rough Bounds)**.** *$7 < L^* < 9$.*

Theorem 22.6 (Spectral Gap from Selection)**.** *$\lambda_1 = \pi^2/(L^*)^2$; conditional on the selection principle, this equals $\dim(G_2)/H^* = 14/99$ (bare topological ratio). The analytical formula gives $\lambda_1 = 6\pi^2/475 \approx 0.12467$.*

Theorem 22.7 (Spectral–Holonomy Principle)**.** *The spectral gap equals holonomy dimension divided by cohomological complexity.*

Theorem 22.8 (Selection Principle Certificate)**.** *Complete selection principle certificate.*

22.2 Cheeger–Buser Inequality

Definition 22.9 (Cheeger Constant)**.** $h(M) := \inf \frac{\mathrm{Area}(\Sigma)}{\min(\mathrm{Vol}(A),\mathrm{Vol}(B))}$ over all separating hypersurfaces Σ.

Theorem 22.10 (K_7 Cheeger Lower Bound)**.** *Cheeger bound: if the bare ratio $14/99$ is used as lower bound, then $\lambda_1 \geq h^2/4 = (14/99)^2/4 = 49/9801$.*

Theorem 22.11 (Mass Gap Exceeds Cheeger)**.** *The bare ratio satisfies $14/99 > (14/99)^2/4 = 49/9801$ (Cheeger-consistent).*

Theorem 22.12 (Cheeger Certificate)**.** *Cheeger–Buser bounds satisfied.*

22.3 KK Spectral Bridge and Yang–Mills

Theorem 22.13 (GIFT Yang–Mills Prediction)**.** *Via the KK spectral bridge (Axiom KK_EFT), the bare algebraic ratio gives* $\Delta_{\mathrm{KK}} = (14/99) \times \Lambda_{\mathrm{QCD}}$.

Theorem 22.14 (Mass Gap in MeV)**.** $\Delta \approx 28.28$ MeV *(with* $\Lambda_{\mathrm{QCD}} = 200$ MeV*).*

Theorem 22.15 (Clay Millennium Candidate)**.** *The KK spectral bridge provides a candidate connection to the Clay Millennium Prize Yang–Mills mass gap problem, via* $\lambda_1 > 0$ *on* K_7 *(one axiom: KK_EFT).*

Theorem 22.16 (Topological Origin)**.** *The algebraic ratio* $\dim(G_2)/H^*$ *has purely topological origin.*

Theorem 22.17 (Yang–Mills Certificate)**.** *Complete KK spectral bridge certificate.*

Chapter 23

Summary and Status

23.1 Proof Status Overview

Module	Theorems	Status
E8 Lattice	15	
G2 Cross Product	10	
Betti Numbers	8	
SO(16) Decomposition	11	
Joyce Theorem	10	
Physical Relations	50+	
TCS Spectral Bounds	15	
Literature Axioms (Langlais/CGN)	8	
Physical Spectral Gap (13/99)	28	(zero axioms)
G_2 Metric Properties	25	
TCS Piecewise Metric	30	
Conformal Rigidity	37	
Spectral Scaling	35	
Poincaré Duality	41	
G_2 Differential Geometry	18	(zero axioms)
Dimensional Hierarchy	25	
Extended Observables	50+	
Mass Gap Ratio & Universal Law	20	
Selection Principle & Yang–Mills	30	
Total	**460+**	–

Theorem 23.1 (Foundations Certificate)**.** *Mathematical infrastructure: E_8 lattice (240 roots), G_2 cross product, G_2 differential geometry (axiom-free), octonion bridge, K_7 Betti numbers, Joyce existence, conformal rigidity, Poincaré duality, G_2 metric properties, TCS piecewise metric.*

Theorem 23.2 (Predictions Certificate)**.** *Published predictions: 33+ dimensionless derivations, 50+ observables, hierarchy solution, exceptional chain, SO(16) decomposition.*

Theorem 23.3 (Spectral Certificate)**.** *Spectral gap programme: bare algebraic ratio* $\dim(G_2)/H^* = 14/99$*, physical ratio* $(\dim(G_2) - h)/H^* = 13/99$*, analytical mass gap* $\lambda_1 = 6\pi^2/475$*, TCS spectral bounds, selection principle, Cheeger inequality, KK spectral bridge.*

Theorem 23.4 (GIFT Grand Certificate)**.** *Complete GIFT framework: 460+ certified relations spanning E_8 lattice, G_2 holonomy, G_2 differential geometry (axiom-free), dimensional hierarchy, 50+ physical observables, spectral theory, bare ratio $14/99$, physical ratio $13/99$, selection principle, KK spectral bridge, Cheeger inequality, G_2 metric properties, TCS piecewise metric, conformal rigidity, and Poincaré duality.*

Structure: three domain-organized pillars — `Foundations.statement` $\wedge$ `Predictions.statement` $\wedge$ `Spectral.statement`*.*

23.2 Key Results

The GIFT framework achieves:

- 0.24% mean deviation across 32 well-measured observables (PDG 2024 / NuFIT 6.0)
- 460+ formally verified relations in Lean 4; 4 logical axioms on the main prediction chain, plus 11 interval-arithmetic certificates for the K3 block of g^*
- Joyce existence theorem for torsion-free G_2 on K_7
- TCS spectral bounds: $\lambda_1 \sim 1/L^2$ (Model Theorem)
- Physical spectral gap $\lambda_1 = 13/99$: axiom-free from Berger classification
- G_2 metric non-flatness via Bieberbach bound ($b_3 = 77 > 35$)
- Spectral degeneracy $[1, 10, 9, 30]$ with topological derivations
- $\det(g) = 65/32$: triple independent derivation (Weyl, cohomological, H^*)
- $\kappa_T^{-1} = 61 = \dim(F_4) + N_{\text{gen}}^2$: structural decomposition
- 38,231× PINN compression: $\dim(\text{SPD}_7) = 28 = 2 \times \dim(G_2)$
- TCS building block asymmetry: $b_3(M_1) - b_3(M_2) = N_{\text{gen}} = 3$
- $H^*(M_1) = \dim(F_4) = 52$: quintic block carries F_4 degrees of freedom
- Matrix space: $7^2 = 2 \cdot \dim(G_2) + b_2 = 28 + 21$
- Conformal rigidity: $28 - 27 - 1 = 0$ residual DOF (G_2 holonomy + determinant)
- $\text{End}(V_7) = 1 \oplus 7 \oplus 14 \oplus 27$ (all four G_2 irreps, $49 = 7^2$)
- Moduli gap: $b_3 - b_2 = 56 = \dim(\text{fund. } E_7) = \text{rank}(E_8) \times \dim(K_7)$
- Poincaré duality: $\beta_{\text{total}} = 198 = 2 \times H^*$ (duality doubles the spectrum)
- Structural identity: $H^* = 1 + 2 \times \dim(K_7)^2$ (manifold dimension controls all)
- Holonomy chain: $\dim(\text{SO}(7)) = b_2 = 21$, $\dim(\text{GL}(7)) = 7^2 = 49$
- Betti–torsion bridge: $b_2 + b_3 = 2 \times (1 + 7 + 14 + 27)$